

On $gsb^\#$ -Continuous and $gsb^\#$ -Irresolute Functions in Topological Spaces

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Abstract

A subset A of a topological space X is said to be generalized semi $b^\#$ -closed set (briefly $gsb^\#$ -closed) if $b^\#cl(A) \leq U$ whenever $A \leq U$ and U is semi open. Using this class of closed sets, the different notion of new class of continuous function called $gsb^\#$ -continuous functions and $gsb^\#$ -irresolute functions in topological spaces are introduced. Basic characterizations and several properties concerning them are also discussed.

Keywords: $b^\#$ -closed set, $gb^\#$ -closed set, $gsb^\#$ -closed set, $gsb^\#$ -continuous maps and $gsb^\#$ -irresolute maps.

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1. Introduction

The generalized closed sets, semi closed, semi-generalized closed sets respectively introduced and studied by Levine (1970), Biswas (1970), Bhattacharyya & Lahiri (1987). The following sets called semi-open, α -open and pre-open are introduced by Levine (1963), Njastad (1965), Mashour et al (1982), respectively. On the other, N. Levine [4] generalized the concepts of closed sets to generalized closed sets. After the works of N. Levine, P. Bhattacharya and P.K. Lahiri [3] introduced the concept of semi-generalized closed set. S.P. Arya and T. Nours [1] defined the generalized semi-open sets and studied some of their properties.

In 2014, Parameswari et al. [6] introduced notions of $b^\#$ -open sets and $b^\#$ -closed sets by taking equality in the definitions of b -open sets and b -closed sets. Vijaya et al. [7] introduced the notion of regular generalized $b^\#$ -closed sets. N. Vithya and P. Thangavelu [8] introduced the class of $gb^\#$ -closed sets. In this paper we shall continue the investigations carried out in [2] and introduce the new class of continuous function namely $gsb^\#$ -continuous maps. Further, we introduce the concept of $gsb^\#$ -irresolute maps and study their properties.

2. Preliminaries

Throughout this paper (X, τ) (or X) denotes a topological space on which no separation axiom is assumed. For any subset A of X , $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A in the topological space X . The following definitions and results are very useful in the subsequent sections.

Definition 2.1: A subset A of a space X is called

- i) α -open if $A \leq int(cl(int(A)))$ and α -closed if $cl(int(cl(A))) \leq A$.
- ii) semi-open if $A \leq cl(int(A))$ and semi-closed if $int(cl(A)) \leq A$.
- iii) generalized closed (briefly g -closed) [9] if $cl(A) \leq U$ whenever $A \leq U$ and U is open.
- iv) g^* -closed [5] if $cl(A) \leq U$ whenever $A \leq U$ and U is g -open.
- v) Generalized $b^\#$ -closed (briefly $gb^\#$ -closed) [8] if $b^\#cl(A) \leq U$ whenever $A \leq U$ and U is open.

- vi) generalized semi closed (briefly gs -closed) if $scl(A) \leq U$ whenever $A \leq U$ and U is open.
- vii) semi generalized closed (briefly sg -closed) if $scl(A) \leq U$ whenever $A \leq U$ and U is semi open.
- viii) Generalized b -closed (briefly gb -closed) if $bcl(A) \leq U$ whenever $A \leq U$ and U is open.
- ix) generalized semi $b^\#$ -closed (briefly $gsb^\#$ -closed) [2] if $b^\#cl(A) \leq U$ whenever $A \leq U$ and U is semi open.

The collection of all semi-closed, α -closed, $b^\#$ -closed, g -closed, g^* -closed, $gb^\#$ -closed, gs -closed, sg -closed, gb -closed sets in topological space (X, r) is denoted by $SC(X)$, $\alpha C(X)$, $b^\#-C(X)$, $g-C(X)$, $g^*-C(X)$, $gb^\#-C(X)$, $gs-C(X)$, $sg-C(X)$, $gb-C(X)$ respectively.

Definition 2.2: A function $f: X \rightarrow Y$ is called

- i) α -continuous if $f^{-1}(V)$ is α -closed in X for each closed set V of Y .
- ii) $b^\#$ -continuous if $f^{-1}(V)$ is $b^\#$ -closed in X for each closed set V of Y .
- iii) g -continuous if $f^{-1}(V)$ is g -closed in X for each closed set V of Y .
- iv) g^* -continuous if $f^{-1}(V)$ is g^* -closed in X for each closed set V of Y .
- v) gb -continuous if $f^{-1}(V)$ is gb -closed in X for each closed set V of Y .
- vi) gs -continuous if $f^{-1}(V)$ is gs -closed in X for each closed set V of Y .
- vii) sg -continuous if $f^{-1}(V)$ is sg -closed in X for each closed set V of Y .
- viii) $gb^\#$ -continuous if $f^{-1}(V)$ is $gb^\#$ -closed in X for each closed set V of Y .

Lemma 2.3[2]: Let (X, r) be a topological space. Then

- i) Every $gsb^\#$ -closed set is $gb^\#$ -closed.
- ii) Every $b^\#$ -closed set is $gsb^\#$ -closed.
- iii) Every $gsb^\#$ -closed set is gs -closed.
- iv) Every $gsb^\#$ -closed set is sg -closed.
- v) Every $gsb^\#$ -closed set is gb -closed.

3. $gsb^\#$ -Continuous maps in Topological Spaces

We introduce $gsb^\#$ -continuous function and its basic properties are studied.

Definition 3.1: A function $f: (X, r) \rightarrow (Y, \sigma)$ is said to be $gsb^\#$ -continuous if $f^{-1}(V)$ is a $gsb^\#$ -closed set of (X, r) for every closed set V of (Y, σ) .

Example 3.2: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. Here $\{b, c\}$ is a closed set of (Y, σ) and $f^{-1}(\{b, c\})$ is a $gsb^\#$ -closed in (X, r) . Therefore, f is a $gsb^\#$ -continuous map.

Theorem 3.3: Every $gsb^\#$ -continuous map is $gb^\#$ -continuous.

Proof. Suppose $f: (X, r) \rightarrow (Y, \sigma)$ is $gsb^\#$ -continuous. Let V be a closed set in (Y, σ) . Since f is $gsb^\#$ -continuous, using definition 3.1, $f^{-1}(V)$ is $gsb^\#$ -closed in X . Again using lemma 2.3(i) and definition 2.2(viii), $f^{-1}(V)$ is $gb^\#$ -closed in X and therefore f is $gb^\#$ -continuous.

Remark 3.4: The converse of the above theorem need not be true as can be seen from the following example.

Example 3.5: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. f is not a $gsb^\#$ -continuous map.

Since $\{b, c\}$ is a closed set of (Y, σ) but $f^{-1}\{b, c\} = \{b, c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is $gb^\#$ -continuous map.

Theorem 3.6: Every $b^\#$ -continuous map is $gsb^\#$ -continuous.

Proof. Suppose $f: (X, r) \rightarrow (Y, \sigma)$ is $b^\#$ -continuous. Let V be a closed set in (Y, σ) . Since f is $b^\#$ -continuous, using definition 2.2(ii), $f^{-1}(V)$ is $b^\#$ -closed in X . Again using lemma 2.3(ii) and definition 3.1, $f^{-1}(V)$ is $gsb^\#$ -closed in X and therefore f is $gsb^\#$ -continuous.

Remark 3.7: The converse of the above theorem need not be true as can be seen from the following example.

Example 3.8: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. f is not a $b^\#$ -continuous map. Since $\{b, c\}$ is a closed set of (Y, σ) but $f^{-1}\{b, c\} = \{b, c\}$ is not a $b^\#$ -closed set of (X, r) . But it is $gsb^\#$ -continuous.

Theorem 3.9: Every $gsb^\#$ -continuous map is gs -continuous.

Proof. Suppose $f: (X, r) \rightarrow (Y, \sigma)$ is $gsb^\#$ -continuous. Let V be a closed set in (Y, σ) . Since f is $gsb^\#$ -continuous, using definition 3.1, $f^{-1}(V)$ is $gsb^\#$ -closed in X . Again using lemma 2.3(iii) and definition 2.2(vi), $f^{-1}(V)$ is gs -closed in X and therefore f is gs -continuous.

Remark 3.10: The converse of the above theorem need not be true as can be seen from the following example.

Example 3.11: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. f is not a $gsb^\#$ -continuous map. Since $\{b, c\}$ is a closed set of (Y, σ) but $f^{-1}\{b, c\} = \{b, c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is gs -continuous map.

Theorem 3.12: Every $gsb^\#$ -continuous map is sg -continuous.

Proof. Suppose $f: (X, r) \rightarrow (Y, \sigma)$ is $gsb^\#$ -continuous. Let V be a closed set in (Y, σ) . Since f is $gsb^\#$ -continuous, using definition 3.1, $f^{-1}(V)$ is $gsb^\#$ -closed in X . Again using lemma 2.3(iv) and definition 2.2(vii), $f^{-1}(V)$ is sg -closed in X and therefore f is sg -continuous.

Remark 3.13: The converse of the above theorem need not be true as can be seen from the following example.

Example 3.14: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. f is not a $gsb^\#$ -continuous map. Since $\{b, c\}$ is a closed set of (Y, σ) but $f^{-1}\{b, c\} = \{b, c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is sg -continuous map.

Theorem 3.15: Every $gsb^\#$ -continuous map is gb -continuous.

Proof. Suppose $f: (X, r) \rightarrow (Y, \sigma)$ is $gsb^\#$ -continuous. Let V be a closed set in (Y, σ) . Since f is $gsb^\#$ -continuous, using definition 3.1, $f^{-1}(V)$ is $gsb^\#$ -closed in X . Again using lemma 2.3(v) and definition 2.2(v), $f^{-1}(V)$ is gb -closed in X and therefore f is gb -continuous.

Remark 3.16: The converse of the above theorem need not be true as can be seen from the following example.

Example 3.17: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. f is not a $gsb^\#$ -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is gb -continuous map.

Remark 3.18: $gsb^\#$ -continuity is independent of α -continuity.

Example 3.19: (i) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. f is not $gsb^\#$ -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is α -continuous map.

(ii) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. f is not α -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{b\}$ is not a α -closed set of (X, r) . But it is $gsb^\#$ -continuous map.

Remark 3.20: $gsb^\#$ -continuity is independent of g -continuity.

Example 3.21: (i) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. f is not $gsb^\#$ -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is g -continuous map.

(ii) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. f is not g -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{b\}$ is not a g -closed set of (X, r) . But it is $gsb^\#$ -continuous map.

Remark 3.22: $gsb^\#$ -continuity is independent of g^* -continuity.

Example 3.23: (i) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = a, f(c) = c$. f is not $gsb^\#$ -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{c\}$ is not a $gsb^\#$ -closed set of (X, r) . But it is g^* -continuous map.

(ii) Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \phi, \{a, b\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = b, f(b) = c, f(c) = a$. f is not g^* -continuous map. Since $\{c\}$ is a closed set of (Y, σ) but $f^{-1}\{c\} = \{b\}$ is not a g^* -closed set of (X, r) . But it is $gsb^\#$ -continuous map.

Remark 3.24: Composition of two $gsb^\#$ -continuous maps need not be $gsb^\#$ -continuous map as seen in the following example.

Example 3.25: Let $X = \{a, b, c\} = Y = Z, r = \{X, \phi, \{a, b\}\}, \sigma = \{Y, \phi, \{a\}, \{b\}, \{a, b\}\}, \tau = \{Z, \phi, \{a, c\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. Define $g: (X, r) \rightarrow (Z, \tau)$ by $g(a) = b, g(b) = c, g(c) = a$. Clearly, f and g are $gsb^\#$ -continuous maps. $g \circ f: (X, r) \rightarrow (Z, \tau)$ is not $gsb^\#$ -continuous. Since $\{b\}$ is a closed set of (Z, τ) but $(g \circ f)^{-1}(\{b\}) = f^{-1}(g^{-1}\{b\}) = f^{-1}(\{a\}) = \{a\}$ is not a $gsb^\#$ -closed set of (X, r) .

Theorem 3.26: Let $f: (X, r) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \tau)$ be any two functions. If g is continuous and f is $gsb^\#$ -continuous, then $g \circ f: (X, r) \rightarrow (Z, \tau)$ is $gsb^\#$ -continuous.

Proof. Suppose $g: Y \rightarrow Z$ is continuous function. Let V be a closed set in Z . Since g is continuous, $g^{-1}(V)$ is closed in Y . Also since $f: X \rightarrow Y$ is $gsb^\#$ -continuous function and using definition 3.1, $f^{-1}(g^{-1}(V))$ is $gsb^\#$ -closed in X . Therefore $g \circ f: X \rightarrow Z$ is $gsb^\#$ -continuous.

4. $gsb^\#$ -Irresolute maps in Topological Spaces

Definition 4.1: A function $f: (X, r) \rightarrow (Y, \sigma)$ is said to be $gsb^\#$ -irresolute if $f^{-1}(V)$ is $gsb^\#$ -closed set of (X, r) for every $gsb^\#$ -closed set V of (Y, σ) .

Example 4.2: Let $X = Y = \{a, b, c\}$ with $r = \{X, \phi, \{b\}, \{a, c\}\}$ and $\sigma = \{Y, \phi, \{a\}\}$. Define $f: (X, r) \rightarrow (Y, \sigma)$ by $f(a) = a, f(b) = b, f(c) = c$. Here $\{b, c\}$ is a $gsb^\#$ -closed set of (Y, σ) and $f^{-1}(\{b, c\})$ is a $gsb^\#$ -closed in (X, r) . Therefore, f is a $gsb^\#$ -irresolute.

Theorem 4.3: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \mathcal{U})$ be any two functions. If g is $gsb^\#$ -irresolute and f is $gsb^\#$ -irresolute, then $g \circ f: (X, \tau) \rightarrow (Z, \mathcal{U})$ is $gsb^\#$ -irresolute.

Proof. Suppose $g: Y \rightarrow Z$ is $gsb^\#$ -irresolute function. Let V be a $gsb^\#$ -closed set in Z . Since g is $gsb^\#$ -irresolute and by definition 4.1, we have, $g^{-1}(V)$ is $gsb^\#$ -closed in Y . Also since $f: X \rightarrow Y$ is $gsb^\#$ -irresolute function and using definition 4.1, $f^{-1}(g^{-1}(V))$ is $gsb^\#$ -closed in X . Therefore, $g \circ f: X \rightarrow Z$ is $gsb^\#$ -irresolute.

Theorem 4.4: Let $f: (X, \tau) \rightarrow (Y, \sigma)$ and $g: (Y, \sigma) \rightarrow (Z, \mathcal{U})$ be any two functions. If g is $gsb^\#$ -continuous and f is $gsb^\#$ -irresolute, then $g \circ f: (X, \tau) \rightarrow (Z, \mathcal{U})$ is $gsb^\#$ -continuous.

Proof. Suppose $g: Y \rightarrow Z$ is $gsb^\#$ -continuous function. Let V be a closed set in Z . Since g is $gsb^\#$ -continuous and by definition 3.1, we have, $g^{-1}(V)$ is $gsb^\#$ -closed in Y . Also since $f: X \rightarrow Y$ is $gsb^\#$ -irresolute function and using definition 4.1, $f^{-1}(g^{-1}(V))$ is $gsb^\#$ -closed in X . Therefore, $g \circ f: X \rightarrow Z$ is $gsb^\#$ -continuous.

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